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FUNCTORIAL STRUCTURE OF UNITS IN A TENSOR PRODUCT

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ABSTRACT. The behavior of units in a tensor product of rings is studied, as one factor varies. For example, let k be an algebraically closed field. Let A and B be reduced rings containing k, having connected spectra. Let $u \in A \otimes_k B$ be a unit. Then $u = a \otimes b$ for some units $a \in A$ and $b \in B$.

Here is a deeper consequence, stated for simplicity in the affine case only. Let k be a field, and let $\varphi : R \to S$ be a homomorphism of finitely generated k-algebras such that $\mathrm{Spec}(\varphi)$ is dominant. Assume that every irreducible component of $\mathrm{Spec}(R_{\mathrm{red}})$ or $\mathrm{Spec}(S_{\mathrm{red}})$ is geometrically integral and has a rational point. Let $B \to C$ be a faithfully flat homomorphism of reduced k-algebras. For A a k-algebra, define $Q(A)$ to be $(S \otimes_k A)^*/(R \otimes_k A)^*$. Then Q satisfies the following sheaf property: the sequence

$$0 \to Q(B) \to Q(C) \to Q(C \otimes_B C)$$

is exact. This and another result are used to prove (5.2) of [7].

1. Introduction

Let k be a field, and let A be a finitely generated k-algebra.[1] We explore the structure of the functor from $\langle\langle k\text{-algebras}\rangle\rangle$ to $\langle\langle\text{abelian groups}\rangle\rangle$ given by $B \mapsto (A \otimes_k B)^*$. More generally, if S is a k-scheme of finite type, not necessarily affine, we study the functor $\mu(S)$ given by $B \mapsto (\Gamma(S, \mathcal{O}_S) \otimes_k B)^*$. This was done in ([8], 4.5) for the case where k is algebraically closed and S is a variety.

We make the assumption that every irreducible component of S_{red} is geometrically integral and has a rational point. We summarize these properties by saying that S is *geometrically stable*. If S is any k-scheme of finite type, we can always find a finite extension k' of k such that $S \times_k \mathrm{Spec}(k')$ is geometrically stable as a k'-scheme.

With the assumption that S is geometrically stable, we find that $\mu(S)$ fits into an exact sequence

$$0 \to \mathbb{G}_m^r \times U \times \mathbb{Z}^n \to \mu(S) \to I \to 0$$

in which I is a sheaf (for the fpqc topology), $I(B) = 0$ for every reduced k-algebra B, and U admits a finite filtration with successive quotients isomorphic to $\mathbb{G}_a^\beta$, for various $\beta \in \mathbb{N} \cup \{\infty\}$. We summarize these properties by saying that I is *nilpotent* and U is *additive*. In the sequence, $\mathbb{Z}^n$ denotes the constant sheaf associated to the

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abelian group $\mathbb{Z}^n$, or equivalently, the functor which represents the constant group scheme associated to the abelian group $\mathbb{Z}^n$.

Moreover, suppose we have a dominant morphism $f : S \to T$, in which both S and T are geometrically stable. There is an induced morphism of functors $\mu(f) : \mu(T) \to \mu(S)$. Let $Q = \mathrm{Coker}[\mu(f)]$. We find that Q also fits into an exact sequence as shown above, except that $\mathbb{G}_m^r \times U \times \mathbb{Z}^n$ is replaced by an extension of a finitely generated abelian group (i.e. the associated constant sheaf) by $\mathbb{G}_m^r \times U$, U is pseudoadditive (see §2), and we do not know if I is a sheaf. Correspondingly, we do not know if Q is a sheaf, but we do know at least that $Q|_{\langle\langle \text{reduced } k\text{-algebras}\rangle\rangle}$ is a sheaf and moreover that the canonical map $Q \to Q^+$ is a monomorphism.

Specializing to the affine case, we see for example that if A is a subalgebra of a k-algebra C (and $\mathrm{Spec}(A)$, $\mathrm{Spec}(C)$ are geometrically stable), then the functor given by $B \mapsto (C \otimes B)^*/(A \otimes B)^*$ fits into such an exact sequence.

We have thus far described the content of the first theorem (4.4) of this paper. Now we describe the second theorem (5.1), which is an application of the first.

Let X be a geometrically stable k-scheme. Let $i : X_0 \to X$ be a nilimmersion, such that the ideal sheaf $\mathcal{N}$ of X_0 in X has square zero. Let P be the functor from $\langle\langle k\text{-algebras}\rangle\rangle$ to $\langle\langle\text{abelian groups}\rangle\rangle$ given by

$$P(B) = \mathrm{Ker}[\mathrm{Pic}(X \times_k \mathrm{Spec}(B)) \to \mathrm{Pic}(X_0 \times_k \mathrm{Spec}(B))].$$

Of course, if X is affine, $P = 0$, but in general P is not zero. We find that P fits into an exact sequence

$$0 \to D \oplus I \to U \to P \to 0,$$

in which I is nilpotent (except possibly not a sheaf), U is pseudoadditive, and D is the constant sheaf associated to a finitely generated abelian group.

Although this theorem does not imply that P is a sheaf, it does imply that if $f : B \to C$ is a faithfully flat homomorphism of reduced k-algebras, then $P(f)$ is injective (5.2). In fact, this holds even if $\mathcal{N}^2 \neq 0$.

We indicate the idea of the proof of the second theorem. We have an exact sequence

$$H^0(X, \mathcal{O}_X^*) \to H^0(X_0, \mathcal{O}_{X_0}^*) \to H^1(X, \mathcal{N}) \to \mathrm{Ker}[\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)] \to 0.$$

Functorializing this yields an exact sequence:

$$0 \to \mathrm{Coker}[\mu(i)] \to \mathbb{G}_a^\beta \to P \to 0,$$

in which $\beta = h^1(X, \mathcal{N})$. The first theorem tells us what $\mathrm{Coker}[\mu(i)]$ is like. The second theorem is deduced from this.

Finally, we describe a theorem about the Picard group, whose proof in [7] uses both theorems of this paper. Let k be a field, and let X be a separated k-scheme of finite type. Then there exists a finite field extension k^+ of k such that for every algebraic extension L of k^+, the canonical map $\mathrm{Pic}(X_L) \to \mathrm{Pic}(X_{L^a})$ is injective.

Acknowledgements. Bob Guralnick supplied the neat proof of (4.1). Faltings kindly provided example (2.6), thereby correcting an error.

Conventions. (a) A *k-functor* is a functor from $\langle\langle k\text{-algebras}\rangle\rangle$ to $\langle\langle\text{sets}\rangle\rangle$. (The usage of the term *k-functor* here is slightly different from the usage in [8].) If V is a k-scheme, then we also let V denote the representable k-functor given by $V(B) = \mathrm{Mor}_{\langle\langle k\text{-schemes}\rangle\rangle}(\mathrm{Spec}(B), V)$.

(b) A k-functor F is a *sheaf* (by which we mean *sheaf for the fpqc topology*) if for every faithfully flat homomorphism $p : B \to C$, the canonical map

$$\Psi_{F,p} : F(B) \to \{x \in F(C) : F(i_1)(x) = F(i_2)(x)\}$$

is bijective, where $i_1, i_2 : C \to C \otimes_B C$ are given by $c \mapsto c \otimes 1$ and $c \mapsto 1 \otimes c$, respectively.

(c) The superscript $+$ is used to denote *associated sheaf.*

(d) If k is a field, X is a k-scheme, and L is a field extension of k, we let X_L denote $X \times_k \operatorname{Spec}(L)$. We let k^a denote an algebraic closure of k.

(e) If X is a scheme, we let $\Gamma(X)$ denote $\Gamma(X, \mathcal{O}_X)$, and we let $\Gamma^*(X)$ denote $\Gamma(X, \mathcal{O}_X)^*$.

(f) If B is a ring, $\operatorname{Nil}(B)$ denotes its nilradical.

(g) k-functors are said to be (*abelian group*)-*valued* if they take values in $\langle\langle$abelian groups$\rangle\rangle$ rather than $\langle\langle$sets$\rangle\rangle$.

We give some definitions which are adapted from [8], pp. 173, 180. If k is a field and X, Y are k-schemes, then $\mathbf{Hom}(X, Y)$ denotes the k-functor given by

$$B \mapsto \operatorname{Mor}_{\langle\langle k\text{-schemes}\rangle\rangle}(X \times_k \operatorname{Spec}(B), Y).$$

An (abelian group)-valued k-functor F is *nilpotent* if it is a sheaf and $F(B) = 0$ for every reduced k-algebra B. We say that F is *subnilpotent* if it can be embedded as a subfunctor of a nilpotent k-functor.

An (abelian group)-valued k-functor is *discrete* if it is a constant sheaf. We also say that such a functor is (for example) *discrete and finitely generated*, if it is the constant sheaf associated to a finitely generated abelian group.

2. Additive k-functors

Let k be a field. We need to consider a countably infinite-dimensional analog of unipotent group schemes over k. Actually, what we will be considering is more restrictive, as we shall only be considering the analog of unipotent group schemes over k which are smooth, connected, and moreover which are k-solvable. (See [10], §5.1.)

The simplest infinite-dimensional example is the (abelian group)-valued k-functor $\mathbb{G}_a^\infty$ given by $B \mapsto \bigoplus_{i=1}^\infty B$. However, this is not good enough for our purposes, since in positive characteristic one can have nontrivial extensions of $\mathbb{G}_a$ by $\mathbb{G}_a$. (See e.g. [13], p. 67, exercise 8 or [12], VII, §2.) We want to define a class of objects which is closed under extension.

Definition. Let k be a field, and let F be an (abelian group)-valued k-functor. Then F is *strictly additive* if it is isomorphic to $\mathbb{G}_a^\alpha$ for some $\alpha \in \{0, 1, \ldots, \infty\}$, and F is *additive* if it admits a filtration:

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = F,$$

whose successive quotients are strictly additive.

This terminology is not perfect, but it is at least consistent with the usage of the word *additive* in [8]: by 3.1(d), if k has characteristic zero, then additive $\Rightarrow$ strictly additive.

We define the *dimension* of an additive k-functor F to be the sum of the dimensions of the successive quotients in a filtration of F, as in the definition of additive. Thus $\dim(F) \in \{0, 1, \ldots, \infty\}$.

If F is additive, we define its *period* to be the smallest n for which there exists a filtration as in the definition of additive.

A direct sum of countably many additive k-functors need not be additive, even if the summands are finite-dimensional. Also, we shall not concern ourselves with uncountable direct sums (e.g. of $\mathbb{G}_a$), as they seem not to arise in practice. The following two statements are easily checked. (Use 3.1(d) for the second one.)

Proposition 2.1. *Let k be a field. Let*

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of (abelian group)-valued k-functors, in which F' and F'' are additive. Then F is additive.

Proposition 2.2. *Let k be a field of characteristic zero. Let*

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of (abelian group)-valued k-functors, in which F', F are additive. Then F'' is additive.

Proposition 2.3. *Let k be a field. Let*

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of (abelian group)-valued k functors, in which F', F are additive and finite-dimensional. Then F'' is additive.

Proof. By ([3], 11.17), $(F'')^+$ is representable. Let $p : F \to (F'')^+$ be the canonical map, which is fpqc-surjective. By ([11], Theorem 10), there exists a morphism $\sigma : (F'')^+ \to F$ of k-functors such that $p \circ \sigma = 1_{(F'')^+}$. Hence $F'' = (F'')^+$. Let X', X, and X'' be the group schemes which represent F', F, and F'', respectively. Then we have an exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\langle\langle$commutative k-group schemes$\rangle\rangle$. Since F is additive and finite-dimensional, X admits a series whose factors are copies of the group scheme $\mathbb{G}_a$. Hence X'' admits a series whose factors are group scheme quotients of $\mathbb{G}_a$. But any quotient of $\mathbb{G}_a$ is 0 or $\mathbb{G}_a$ ([10], 2.3), so X'' admits a series whose factors are the group scheme $\mathbb{G}_a$. From the argument at the beginning of the proof (showing that under certain circumstances fpqc-surjective $\Rightarrow$ surjective), we see that F'' admits a series whose factors are the (abelian group)-valued k-functor $\mathbb{G}_a$. Hence F'' is additive. □

Unfortunately, (2.2) fails in positive characteristic. We will give an example of this, but there are a couple of preliminaries:

Lemma 2.4. *If F and G are strictly additive and $\pi : F \to G$ is an epimorphism in $\langle\langle$(abelian group)-valued k-functors$\rangle\rangle$, then π splits.*

Sketch. The lemma is clear if $\text{char}(k) = 0$, so we may assume that $\text{char}(k) = p > 0$. We do the case where $F = G = \mathbb{G}_a^\infty$; the proof in the other cases is the same. Let e_i denote the element of $\mathbb{G}_a^\infty(k)$ which has 1 in the ith spot and 0's elsewhere. Let $A = k[t]$. For each i, choose $f_i \in \mathbb{G}_a^\infty(A)$ such that $\pi(f_i) = te_i$. Let g_i be the part of f_i involving only the monomials $t, t^p, t^{p^2}, \ldots$. Since the monomials which appear in an expression for π also have this form (with various t), it follows that $\pi(g_i) = te_i$.

Regard g_i as a function of t. Define $\sigma : G \to F$ by $\sigma(be_i) = g_i(b)$, where B is a k-algebra and $b \in B$. Then σ splits π. □

Corollary 2.5. *If F is strictly additive and G is additive, and $\pi : F \to G$ is an epimorphism in $\langle\langle$(abelian group)-valued k-functors$\rangle\rangle$, then G is strictly additive.*

Proof. Induct on the period of G. If $\mathrm{period}(G) \le 1$ we are done. Otherwise, we can find $G' \subset G$ and an exact sequence

$$0 \to G' \to G \xrightarrow{p} H \to 0$$

in which H is strictly additive, G' is additive, and $\mathrm{period}(G') < \mathrm{period}(G)$. By (2.4), $p \circ \pi$ splits. Hence p splits. Hence $\mathrm{period}(G) = \mathrm{period}(G')$: contradiction. □

Example 2.6 (provided by G. Faltings)**.** Let $f : \mathbb{G}_a^\infty \to \mathbb{G}_a^\infty$ be given by

$$(x_1, x_2, x_3, \dots) \mapsto (x_1, x_2 - x_1^p, x_3 - x_2^p, \dots).$$

Then f is a monomorphism. Let $F'' = \mathrm{Coker}(f)$. If F'' were additive, then by (2.5) F'' would be strictly additive, and so by (2.4) f would split. However, this is clearly not the case. Hence F'' is not additive.

The following generalization of *additive* allows us to work around the behavior illustrated by the example:

Definition. An (abelian group)-valued k-functor P is *pseudoadditive* if for some $n \in \mathbb{N}$ there exists an exact sequence

$$0 \to U_1 \to U_2 \to \cdots \to U_n \to P \to 0$$

in $\langle\langle$(abelian group)-valued k-functors$\rangle\rangle$ in which $U_1, \dots, U_n$ are additive.

By example (2.6), one cannot always take $n = 1$, i.e. additive $\neq$ pseudoadditive. We do not know if one can always take $n = 2$.

Proposition 2.7. *If P is pseudoadditive, then P is a sheaf.*

Proof. Let $\mathcal{C}$ be the class of (abelian group)-valued k-functors F with the property that for any faithfully flat homomorphism $B \to C$ of k-algebras, the usual Čech complex

$$0 \to F(B) \to F(C) \to F(C \otimes_B C) \to F(C \otimes_B C \otimes_B C) \to \cdots$$

is exact. Then $\mathbb{G}_a^\alpha \in \mathcal{C}$ for all α. If

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence and any two of F', F, F'' are in $\mathcal{C}$, then so is the third. Hence P is in $\mathcal{C}$, so P is a sheaf. □

3. Extensions in $\langle\langle$(abelian group)-valued k-functors$\rangle\rangle$

For any objects F_1, F_2 in an abelian category, one can define an abelian group $\mathrm{Ext}^1(F_1, F_2)$, whose elements are isomorphism classes of extensions

$$0 \to F_2 \to F \to F_1 \to 0.$$

(The general theory is described in [9], Chapter VII, among other places.) Also, we will refer to such an exact sequence as defining an *extension of F_1 by F_2.*

In particular, the theory applies to $\langle\langle$(abelian group)-valued k-functors$\rangle\rangle$. We shall say that an exact sequence

$$0 \to F_2 \to F \xrightarrow{\pi} F_1 \to 0$$

in this category is *set-theoretically split* if there exists a morphism of k-functors $\sigma : F_1 \to F$ such that $\pi \circ \sigma = 1_{F_1}$. We also refer to *set-theoretically split extensions*. Let $\mathrm{Ext}^1_s(F_1, F_2)$ denote the subgroup of elements of $\mathrm{Ext}^1(F_1, F_2)$ which correspond to set-theoretically split extensions.

To compute $\mathrm{Ext}^1_s(F_1, F_2)$, we copy (with appropriate but minor changes) some definitions which may be found in ([12], VII, §4). For this discussion, fix F_1 and F_2. A *symmetric factor system* is a morphism $f : F_1 \times F_1 \to F_2$ of k-functors such that

$$0 = f(y, z) - f(x + y, z) + f(x, y + z) - f(x, y),$$

$$f(x, y) = f(y, x)$$

for all k-algebras B and all $x, y, z \in F_1(B)$. If $g : F_1 \to F_2$ is a morphism of k-functors, then there is a symmetric factor system δg defined by

$$\delta g(x, y) = g(x + y) - g(x) - g(y);$$

such a system is called *trivial*. The group structure on F_2 makes the set of symmetric factor systems into a group. Then by standard arguments, $\mathrm{Ext}^1_s(F_1, F_2)$ is isomorphic to the group of symmetric factor systems, modulo the subgroup of trivial factor systems.

If F_1 and F_2 are sheaves, then one can also compute the group $\mathrm{Ext}^1_{\mathrm{fpqc}}(F_1, F_2)$, i.e. the group of isomorphism classes of extensions of F_1 by F_2 in $\langle\langle$(abelian group)-valued k-functors which are sheaves$\rangle\rangle$. If moreover F_1 and F_2 are represented by commutative group schemes X_1 and X_2 of finite type over k, then (see [4], 5.4 and [2], 3.5, 7.3(ii)) $\mathrm{Ext}^1_{\mathrm{fpqc}}(F_1, F_2) = \mathrm{Ext}^1(X_1, X_2)$, where the latter Ext group is computed relative to the abelian category $\langle\langle$commutative group schemes of finite type over $k\rangle\rangle$. For arbitrary sheaves F_1, F_2, we have $\mathrm{Ext}^1(F_1, F_2) \subset \mathrm{Ext}^1_{\mathrm{fpqc}}(F_1, F_2)$, but not equality in general, as may be seen e.g. from the exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \overset{t \mapsto t^p - t}{\to} \mathbb{G}_a \to 0$$

in the group scheme category, where $\mathrm{char}(k) = p > 0$.

Proposition 3.1. *Let k be a field. We consider objects and morphisms in $\langle\langle$(abelian group)-valued k-functors$\rangle\rangle$. Then*:

(a) *If $W \to G$ is an epimorphism, and V is additive, then the induced map* $\mathrm{Mor}_{\langle\langle k\text{-functors}\rangle\rangle}(V, W) \to \mathrm{Mor}_{\langle\langle k\text{-functors}\rangle\rangle}(V, G)$ *is surjective.*

(b) *If V is additive, then* $\mathrm{Ext}^1_s(V, F) = \mathrm{Ext}^1(V, F)$ *for all F.*

(c) *If F is a sheaf then* $\mathrm{Ext}^1(\mathbb{Z}, F) = 0$.

(d) $\mathrm{Ext}^1(U, V) = 0$ *if U and V are additive and k has characteristic zero. Hence in characteristic zero, additive $\Rightarrow$ strictly additive.*

(e) *If we have an exact sequence*

$$0 \to U_1 \to U_2 \to P \to 0$$

in which U_1 and U_2 are additive, then $\mathrm{Ext}^1(\mathbb{G}_m, P) = 0$.

(f) $\mathrm{Ext}^1(I, D) = 0$ *if I is subnilpotent and D is discrete.*

Proof. (a) First we prove this when $V = \mathbb{G}_a^\alpha$ for some α. Suppose that $\alpha < \infty$. Then for all X, we have $\mathrm{Mor}_{\langle\langle k\text{-functors}\rangle\rangle}(\mathbb{G}_a^\alpha, X) = X(k[t_1, \ldots, t_\alpha])$ by the Yoneda Lemma, and so the claim follows if $\alpha < \infty$.

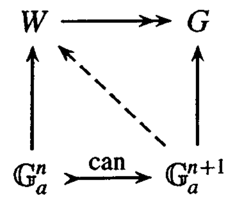

FIGURE 1

Now suppose that $\alpha = \infty$. Then the essential point is that in a commutative diagram one can fill in a dotted arrow as shown in Figure 1. To see this, consider the k-algebra maps $\pi : k[t_1, \ldots, t_{n+1}] \to k[t_1, \ldots, t_n]$ and $i : k[t_1, \ldots, t_n] \to k[t_1, \ldots, t_{n+1}]$, given in both cases by sending t_i to t_i for $i \leq n$, and by $t_{n+1} \mapsto 0$, in the first case. We apply both W and G to these morphisms. Let η be the given morphism from W to G. Now suppose we are given $r \in G(k[t_1, \ldots, t_{n+1}])$ and $s \in W(k[t_1, \ldots, t_n])$, mapping to the same element of $G(k[t_1, \ldots, t_n])$. To get the needed dotted arrow, we have to lift to a common element $u \in W(k[t_1, \ldots, t_{n+1}])$. First find $u_0 \in W(k[t_1, \ldots, t_{n+1}])$ which maps to r. Let s_0 be the image of u_0 in $W(k[t_1, \ldots, t_n])$. Then $s - s_0$ maps to zero in $G(k[t_1, \ldots, t_n])$. Let $u = W(i)(s - s_0) + u_0$. Then $W(\pi)(u) = s$ and

$$\begin{aligned}\eta(u) &= \eta(W(i)(s - s_0)) + \eta(u_0)\\ &= G(i)(\eta(s - s_0)) + r\\ &= G(i)(0) + r = r,\end{aligned}$$

as required. Thus the dotted arrow exists.

Now suppose that V is arbitrary. From what we have just shown, it follows that $\mathrm{Ext}^1_s(\mathbb{G}_a^\alpha, F) = \mathrm{Ext}^1(\mathbb{G}_a^\alpha, F)$ for all F. But in a set-theoretically split exact sequence, the middle term is isomorphic as a k-functor to the product of the end terms. It follows that $V \cong \mathbb{G}_a^\beta$ in $\langle\langle k\text{-functors}\rangle\rangle$, for some β. Hence (a) holds when V is arbitrary.

(b) follows immediately from (a).

(c) We have to show that if $\pi : H \to \mathbb{Z}$ is an epimorphism in $\langle\langle$(abelian group)-valued k-functors $\rangle\rangle$, and H is a sheaf, then π splits. For each $n \in \mathbb{Z}$, let $y_n \in \mathbb{Z}(k)$ correspond to the constant map $\mathrm{Spec}(k) \to \mathbb{Z}$ of topological spaces with value n, and choose $x_1 \in H(k)$ such that $\pi(x_1) = y_1$. For each $n \in \mathbb{Z}$, define $x_n \in H(k)$ to be nx_1. Define $\sigma : \mathbb{Z} \to H$ as follows. For any ring B, an element $\lambda \in \mathbb{Z}(B)$ corresponds to a locally constant map $\mathrm{Spec}(B) \to \mathbb{Z}$ of topological spaces, and therefore we may write $B = B_1 \times \cdots \times B_n$ in such a way that λ is induced by $(y_{r_1}, \ldots, y_{r_n})$ for suitable $r_1, \ldots, r_n \in \mathbb{Z}$. Since H is a sheaf for the Zariski topology, there is a unique element $x_{r_1,\ldots,r_n} \in H(k^n)$ whose image in $H(k)$ under the ith projection map is x_{r_i}. Now set $\sigma(\lambda)$ equal to the image of $x_{r_1,\ldots,r_n}$ under the canonical map $H(k^n) \to H(B_1 \times \cdots \times B_n)$. This defines σ, and thus proves that π splits.

(d) It suffices to show that $\mathrm{Ext}^1(\mathbb{G}_a^\alpha, \mathbb{G}_a^\beta) = 0$ for all α, β. Moreover, since Ext^1 converts a coproduct in the first variable into a product, we may assume that $\alpha = 1$. By (b), it suffices to show that $\mathrm{Ext}^1_s(\mathbb{G}_a, \mathbb{G}_a^\beta) = 0$. Arguing as in the proof of (e), below, it suffices to show that $\mathrm{Ext}^1_s(\mathbb{G}_a, \mathbb{G}_a^n) = 0$, and moreover we may as well take $n = 1$. Suppose we have an exact sequence

$$0 \to \mathbb{G}_a \to X \to \mathbb{G}_a \to 0.$$

By ([10], 3.9 ter.), $X \cong \mathbb{G}_a^2$. But (in characteristic zero) morphisms from $\mathbb{G}_a^n$ to $\mathbb{G}_a^m$ are in bijective correspondence with vector space homomorphisms from k^n to k^m, so the sequence splits.

(e) First suppose that $\operatorname{char}(k) = 0$. Then $P \cong \mathbb{G}_a^\alpha$ for some α. If $\alpha < \infty$, the statement follows from [[10], 5.1.1(i)]. We have $\operatorname{Ext}^1(\mathbb{G}_m, \mathbb{G}_a^\infty) = \operatorname{Ext}_s^1(\mathbb{G}_m, \mathbb{G}_a^\infty)$ since $\mathbb{G}_m$ is representable. Therefore an extension of $\mathbb{G}_m$ by $\mathbb{G}_a^\infty$ corresponds to a symmetric factor system $f : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_a^\infty$. For $n \gg 0$, we can find a morphism $f_n : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_a^n$ through which f factors. But then f_n is a symmetric factor system, and so f_n is trivial, since we already know that $\operatorname{Ext}^1(\mathbb{G}_m, \mathbb{G}_a^n) = 0$. Hence f is trivial. Hence $\operatorname{Ext}^1(\mathbb{G}_m, \mathbb{G}_a^\infty) = 0$.

Now suppose that k has characteristic $p > 0$. Since $\mathbb{G}_m$ and P are sheaves, it suffices to show that $\operatorname{Ext}^1_{\mathrm{fpqc}}(\mathbb{G}_m, P) = 0$. For n sufficiently large, multiplication by p^n gives a zero map from P to P. It follows from the fpqc-exact sequence

$$1 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 1$$

that is enough to show $\operatorname{Hom}(\mu_{p^n}, P) = 0$. Let $f : \mu_{p^n} \to P$ be a morphism. Let H be the fiber product of μ_{p^n} and U_2 over P. Then we have an exact sequence:

$$(*) \qquad 0 \to U_1 \to H \to \mu_{p^n} \to 1.$$

We will show that this sequence splits. We can do this by showing that $\operatorname{Ext}^1(\mu_{p^n}, U_1) = 0$, but by the definition of additive, it is clearly enough to show that $\operatorname{Ext}^1(\mu_{p^n}, \mathbb{G}_a^\alpha) = 0$ for all α. Arguing as in the characteristic zero case, one sees further that it is enough to show that $\operatorname{Ext}^1(\mu_{p^n}, \mathbb{G}_a) = 0$. This is a special case of [10], 5.1.1(d). Hence $(*)$ splits. Hence there exists a morphism $\sigma : \mu_{p^n} \to U_2$ such that $\pi \circ \sigma = f$, where $\pi : U_2 \to P$ is the given map. Now I claim that $\sigma = 0$. For this (arguing as above), it is enough to show that $\operatorname{Hom}(\mu_{p^n}, \mathbb{G}_a) = 0$. It is enough to do this when $k = k^a$, and then the statement is well-known. Hence $f = 0$. Hence $\operatorname{Hom}(\mu_{p^n}, P) = 0$, which completes the proof.

(f) Let

$$0 \to D \to L \to I \to 0$$

be an exact sequence of (abelian group)-valued k-functors. Define a k-functor I' by $I'(B) = \operatorname{Ker}[L(B) \to L(B_{\mathrm{red}})]$. Then I' defines a splitting of the sequence. □

4. Functorial structure of units in a tensor product

The main purpose of this section is to prove (4.4), which generalizes ([8], 4.5). The preparatory lemmas are similar to those in ([8], §4), and we shall omit their proofs if the proofs of the corresponding statements in [8] carry over with minor changes.

Lemma 4.1. *Let k be an algebraically closed field. Let A and B be reduced rings containing k, having connected spectra. Let $u \in A \otimes_k B$ be a unit. Then $u = a \otimes b$ for some units $a \in A$ and $b \in B$.*

Proof. The statement generalizes ([8], 4.2), but we give a new and simpler proof, due to Guralnick. First, we may assume that A and B are finitely generated as k-algebras.

Let X be the set of maximal ideals of A, and let Y be the set of maximal ideals of B. Let $x_0 \in X$, $y_0 \in Y$. We will prove the lemma by showing that for all $x \in X$,

$y \in Y$, we have

$$(*) \qquad u(x, y) = \frac{u(x, y_0)u(x_0, y)}{u(x_0, y_0)}.$$

For this we may suppose that k is uncountable. By a variant of a result of Roquette (see [7], 1.6) the group B^*/k^* is finitely generated, so $F = \{f \in B^* : f(y_0) = 1\}$ is countable. For each $f \in F$, let

$$Q(f) = \{x \in X : u(x, y) = u(x, y_0)f(y) \text{ for all } y \in Y\}.$$

Then $Q(f)$ is a closed subset of X. For any given $x \in X$, the function on Y given by $y \mapsto u(x, y)/u(x, y_0)$ sends y_0 to 1 and so lies in F. Hence $X = \bigcup_{f\in F} Q(f)$.

Since k is uncountable, it follows that if I is an irreducible component of X, then $I \subset Q(f)$ for some $f \in F$. Hence for any fixed $y \in Y$, the function $g_y : X \to k$ given by $x \mapsto u(x, y)/u(x, y_0)$ is constant on each irreducible component of X. Since X is connected, g_y is constant. Then $u(x, y) = u(x, y_0)g_y(x_0)$, which proves $(*)$. □

Corollary 4.2. *Let k be an algebraically closed field. Let A and B be rings containing k, having connected spectra. Assume that A is reduced. Let $\mathfrak{m} \subset A$ be a maximal ideal such that $A/\mathfrak{m} = k$. Then $(A \otimes_k B)^*$ is the direct sum of the two subgroups A^*B^* and $1 + (\mathfrak{m} \otimes \mathrm{Nil}(B))$.* [2]

Corollary 4.3. *Let k be an algebraically closed field. Let A and B be rings containing k. Assume that A is reduced and has a connected spectrum. Let $\mathfrak{m} \subset A$ be a maximal ideal such that $A/\mathfrak{m} = k$.*

- *For any decomposition $B = B_1 \times \cdots \times B_n$, there is a subgroup:*

$$\mu(B_1, \ldots, B_n) = \bigoplus_{i=1}^{n} [A^*B_i^* \oplus (1 + (\mathfrak{m} \otimes \mathrm{Nil}(B_i)))]$$

  of $(A \otimes_k B)^*$.
- *For any $x \in (A \otimes_k B)^*$, there exists a decomposition $B = B_1 \times \cdots \times B_n$ such that $x \in \mu(B_1, \ldots, B_n)$.*
- *If B has only finitely many idempotent elements (e.g. if B is noetherian), we can write $B = B_1 \times \cdots \times B_n$ for rings B_i having connected spectra. Then $\mu(B_1, \ldots, B_n) = (A \otimes_k B)^*$.*

Let k be a field, and let S be a k-scheme of finite type. Let $F = \mathbf{Hom}(S, \mathbb{G}_m)$. Then

$$F(B) = \Gamma(S \times \mathrm{Spec}(B), \mathcal{O}^*_{S\times\mathrm{Spec}(B)}) = [\Gamma(S, \mathcal{O}_S) \otimes_k B]^*,$$

by ([6], 9.3.13(i)). In particular, if $S = \mathrm{Spec}(A)$, then $F(B) = (A \otimes_k B)^*$. The next theorem gives an abstract description of F, and thus (in effect) a description of how units in a tensor product $A \otimes_k B$ vary as B varies. First, for convenience, we encapsulate the following definition:

Definition. Let k be a field. A k-scheme S is *geometrically stable* if (1) it is of finite type, and (2) every irreducible component of S_{red} is geometrically integral and has a rational point.

---

[2] Statements 4.3 and 4.4 from [8] should also have the hypothesis that $A/\mathfrak{m} = k$.

Theorem 4.4. *Let k be a field. Define an* (*abelian group*)*-valued k-functor F to be of type* ($*$) *if there exist exact sequences*

$$(\dagger) \qquad 0 \to R \to F \to I \to 0,$$

$$(\dagger\dagger) \qquad 0 \to \mathbb{G}_m^r \times U \to R \to L \to 0$$

in $\langle\langle$(*abelian group*)*-valued k-functors$\rangle\rangle$, in which $r \geq 0$, U is pseudoadditive, I is subnilpotent, and L is discrete and finitely generated.*

(a) *Let S be a geometrically stable k-scheme. Then $\mathbf{Hom}(S, \mathbb{G}_m)$ is of type* ($*$) *and we have $r =$ the number of connected components of S. Also U is additive of dimension* $\dim_k \mathrm{Nil}[\Gamma(S, \mathcal{O}_S)]$. *Also, I is nilpotent, L is free, and* ($\dagger\dagger$) *splits.*

(b) *Let S and T be geometrically stable k-schemes, and let $f : S \to T$ be a dominant morphism of k-schemes. Then the cokernel of* $\mathbf{Hom}(f, \mathbb{G}_m)$ *is of type* ($*$). *Moreover, r equals the number of connected components of S minus the number of connected components of T.*

Corollary 4.5. *Let S and T be geometrically stable k-schemes, and let $f : S \to T$ be a dominant morphism of k-schemes. Let Q be the cokernel of* $\mathbf{Hom}(f, \mathbb{G}_m)$. *Then the canonical map $Q \to Q^+$ is a monomorphism, and $Q|_{\langle\langle reduced\ k\text{-}algebras\rangle\rangle}$ is a sheaf, in the sense that if $p : B \to C$ is a faithfully flat homomorphism of reduced k-algebras, then $\psi_{Q,p}$ (see §1, convention* (b)) *is bijective.*

Remarks. (i) In part (a) of the theorem, one can choose ($\dagger$) so that it splits if S is reduced but probably not in general.

(ii) In part (b), the sequence ($\dagger\dagger$) does not always split. For an example, take k to be an imperfect field of characteristic p, let $u \in k - k^p$, and let f be Spec of the ring map $k[t, t^{-1}] \to k[x, x^{-1}] \times k$, given by $t \mapsto (x^p, u)$.

(iii) The hypothesis that the schemes in the theorem be geometrically stable can be weakened slightly, as is indicated in the proof. They presumably can be weakened further, but we do not know what is possible in this direction.

(iv) We suspect that I in part (b) of the theorem is a sheaf (and thus satisfies the definition of *nilpotent*). If true, this would imply (in the corollary) that the cokernel of $\mathbf{Hom}(f, \mathbb{G}_m)$ is a sheaf. To prove that I is a sheaf, it would be sufficient (at least in the case where S and T are connected) to show that if C is a subalgebra of a reduced k-algebra A, then the (abelian group)-valued k-functor given by

$$B \mapsto \frac{1 + A \otimes \mathrm{Nil}(B)}{1 + C \otimes \mathrm{Nil}(B)}$$

is a sheaf.

(v) In part (b), we have U pseudoadditive with $n = 2$, as in the definition of pseudoadditive. However, it is conceivable that U is always additive.

Proof of (4.4). The hypothesis that S be geometrically stable is chosen for simplicity and we note here some consequences which are in fact sufficient to prove the theorem:

(A) every connected component of S has a rational point;

(B) S_{red} is geometrically reduced [by [5], 4.6.1(e)]. Moreover, (A) and (B) also imply:

(C) every connected component of S is geometrically connected by [by [5], 4.5.14];

(D) if Q is a connected component of S, then k is integrally closed in $\Gamma(Q_{\mathrm{red}})$. All of these comments apply equally to T.

Now we want to reduce to the affine case. This is not literally possible, because $\Gamma(S)$ need not be finitely generated as a k-algebra. What we can do is reformulate the theorem in terms of a certain class of k-algebras. This class is chosen simply to serve the needs of the proof: a k-algebra is *good* if it is of the form $\Gamma(S)/N$, where S is a geometrically stable k-scheme and $N \subset \Gamma(S)$ is a nilpotent ideal. Here is a reformulation of the theorem in terms of good k-algebras:

(a) Let A be a good k-algebra. Then $B \mapsto (A \otimes B)^*$ is of type $(*)$ and we have $r =$ the number of connected components of $\operatorname{Spec}(A)$, $\dim(U) = \dim_k \operatorname{Nil}(A)$. Also, L is free and I is nilpotent.

(b) Let $\phi : C \to A$ be a homomorphism of good k-algebras. Assume that $\operatorname{Ker}(\phi)$ is nilpotent. Then the cokernel of the morphism from $B \mapsto (C \otimes B)^*$ to $B \mapsto (A \otimes B)^*$ is of type $(*)$. Moreover, r equals the number of connected components of $\operatorname{Spec}(A)$ minus the number of connected components of $\operatorname{Spec}(C)$.

Since the map $(C \otimes B)^* \to (C/\operatorname{Ker}(\phi) \otimes B)^*$ is surjective, we may reduce to the case where ϕ is *injective*. It was the need for this reduction which led to the introduction of good k-algebras in the proof.

We proceed to build a diagram involving (abelian group)-valued k-functors, which we associate to A, and which is functorial in A.

Let G_A be given by $G_A(B) = (A_{\mathrm{red}} \otimes B)^*$. Write $A_{\mathrm{red}} = A_1 \times \cdots \times A_r$, where $A_1, \ldots, A_r$ have connected spectra. We can identify $A \otimes B$ with $(A_1 \otimes B) \times \cdots \times (A_r \otimes B)$. Let F_A be the sheaf associated to the subfunctor of G_A given by $B \mapsto \{(a_1 \otimes b_1, \ldots, a_r \otimes b_r) : a_i \in A_i^*, b_i \in B^*\}$. Let E_A be the subfunctor of F_A given by $E_A(B) = \{(b_1, \ldots, b_r) : b_1, \ldots, b_r \in B^*\}$. Let $D_A = F_A/E_A$. Let $I_A = G_A/F_A$.

Define H_A by $H_A(B) = (A \otimes B)^*$. Let $p : H_A \to G_A$ be the canonical map, and let U_A be its kernel. We have $U_A(B) = 1 + \operatorname{Nil}(A) \otimes B$. For each $n \in \mathbb{N}$, let U_A^n be given by $U_A^n(B) = 1 + \operatorname{Nil}(A)^n \otimes B$.

Here is the diagram of (abelian group)-valued k-functors which we have built:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & & U_A & = & U_A^1 & \supset & U_A^2 \;\supset \cdots \\
 & & \downarrow & & \downarrow & & & & \\
 & & E_A & & H_A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F_A & \to & G_A & \to & I_A & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & D_A & & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

We proceed to analyze the various components of this diagram. In particular, we will show that (1) $G_A \cong F_A \times I_A$ and I_A is nilpotent, (2) $E_A \cong \mathbb{G}_m^r$, where r is the number of connected components of $\operatorname{Spec}(A)$, (3) D_A is represented by a constant group scheme (corresponding to a finitely generated free abelian group), and that (4) U_A is additive. For these purposes, we may assume that $\operatorname{Spec}(A)$ is connected. Then $E_A(B) = B^*$.

(1) Let $\mathfrak{m}_A \subset A_{\mathrm{red}}$ be a maximal ideal such that $A/\mathfrak{m}_A = k$. (This is possible by (A).) Define a subfunctor I'_A of G_A by $I'_A(B) = 1 + \mathfrak{m}_A \otimes \operatorname{Nil}(B)$. Let $\psi : F_A \oplus I'_A \to G_A$ be the canonical map. We will show that ψ is an isomorphism. If k is algebraically closed, this follows from (4.3). But both the source and the target

of ψ are sheaves, so it follows (using (B) and (C)) that ψ is an isomorphism for any k. Thus $I'_A \cong I_A$, so I_A is nilpotent.

(2) We have $E_A \cong \mathbb{G}_m$.

(3) Write $A = \Gamma(S)/N$, as in the definition of good. Let $\mathcal{N}$ be the nilradical of S. Then we have an exact sequence:

$$0 \to H^0(S, \mathcal{N}) \to H^0(S, \mathcal{O}_S) \to H^0(S_{\mathrm{red}}, \mathcal{O}_{S_{\mathrm{red}}})$$

and so A_{red} is a subring of $H^0(S_{\mathrm{red}}, \mathcal{O}_{S_{\mathrm{red}}})$. Since by (D) k is integrally closed in $H^0(S_{\mathrm{red}}, \mathcal{O}_{S_{\mathrm{red}}})$, it follows by a version of Roquette's Theorem ([7], 1.6) that $D_A(k)$ is free abelian of finite rank. In fact, D_A is represented by the corresponding constant group scheme.

(4) For each n we have

$$\frac{U_A^n}{U_A^{n+1}}(B) = \frac{1 + \mathrm{Nil}(A)^n \otimes B}{1 + \mathrm{Nil}(A)^{n+1} \otimes B} \cong \frac{\mathrm{Nil}(A)^n}{\mathrm{Nil}(A)^{n+1}} \otimes B$$

as (abelian group)-valued k-functors. Hence U_A is additive.

Now we describe H_A, making a number of noncanonical choices. We have $F_A \cong E_A \times D_A$ noncanonically, e.g. by 3.1(c), but it is easily proved directly. Hence $F_A \cong \mathbb{G}_m^r \times \mathbb{Z}^n$ for some n. Also, we have shown that $G_A \cong F_A \times I_A$. Therefore we have an exact sequence

$$0 \to U_A \to H_A \xrightarrow{q} \mathbb{G}_m^r \times \mathbb{Z}^n \times I_A \to 0.$$

Let $M = q^{-1}(\mathbb{G}_m^r \times \mathbb{Z}^n)$. Then we have an exact sequence

$$0 \to U_A \to M \to \mathbb{G}_m^r \times \mathbb{Z}^n \to 0.$$

By 3.1(c,e) this sequence splits. This proves (a).

Now, to prepare for proving (b), we analyze the functorial behavior of each basic component of the big diagram shown above. Let $\phi : C \to A$ be an injective homomorphism of good k-algebras.

First we analyze E_ϕ. It is a monomorphism, corresponding to a map $\mathbb{G}_m^{r_1} \to \mathbb{G}_m^{r_2}$, for some r_1 and r_2, which is given by an $r_2 \times r_1$ matrix of 0's and 1's. The cokernel of E_ϕ is isomorphic to $\mathbb{G}_m^{r_2 - r_1}$.

Now we analyze D_ϕ. Let us show that D_ϕ is a monomorphism. Since its source and target are constant sheaves, it suffices to show that $D_\phi(k^a)$ is injective. The assertion then boils down to showing that if one has a dominant morphism $\psi : V \to W$ of reduced schemes of finite type over an algebraically closed field k, and W is connected, and $g : W(k) \to k$ is a nonconstant regular function, then $g \circ \psi(k)$ is not constant on each connected component of V. This is clear, so D_ϕ is a monomorphism. The cokernel of D_ϕ is the constant sheaf associated to a finitely generated abelian group.

We show that I_ϕ is a monomorphism. In the process of doing so, we justify remark (iv) after (4.5). Also, once we know that I_ϕ is a monomorphism, it will follow immediately that $\mathrm{Coker}(I_\phi)$ is subnilpotent. We may assume that $\mathrm{Spec}(C)$ is connected.

We have a canonical map $1 + C_{\mathrm{red}} \otimes \mathrm{Nil}(B) \to I_C(B)$, and likewise for A. From our discussion of I', it is clear that these maps are surjective. Letting $X_C(B)$ and

$X_A(B)$ denote their kernels, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \to & X_C(B) & \to & 1+C_{\mathrm{red}}\otimes \mathrm{Nil}(B) & \to & I_C(B) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & X_A(B) & \to & 1+A_{\mathrm{red}}\otimes \mathrm{Nil}(B) & \to & I_A(B) & \to & 1 \end{array}$$

We describe $X_C(B)$. Let $x \in X_C(B)$. Locally (for the fpqc topology) on B, we may write $x = c \otimes b = 1 + \sum_{i=1}^{s} c_i \otimes b_i$, where $c \in C^*_{\mathrm{red}}, b \in B^*, c_i \in C_{\mathrm{red}}$, and $b_i \in \mathrm{Nil}(B)$. It follows that b must lie in the k-linear span of 1 and the b_i, and so (after adjusting c) we may assume that $b = 1 + n$, where $n \in \mathrm{Nil}(B)$. Passing to $C_{\mathrm{red}} \otimes B_{\mathrm{red}}$, we see then that $c = 1$. Hence $x = 1 + n$. From this it follows that $X_C(B) = 1 + \mathrm{Nil}(B)$.

Similarly, we have $X_A(B) = (1 + \mathrm{Nil}(B)) \times \cdots \times (1 + \mathrm{Nil}(B))$, with one copy for each connected component of $\mathrm{Spec}(A)$. It follows (details omitted) that the canonical map

$$\frac{X_A(B)}{X_C(B)} \to \frac{1 + A_{\mathrm{red}} \otimes \mathrm{Nil}(B)}{1 + C_{\mathrm{red}} \otimes \mathrm{Nil}(B)}$$

is injective, and hence that I_ϕ is a monomorphism.

Assume now that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(C)$ are connected. Let $\mathfrak{m}_C$ be the preimage of $\mathfrak{m}_A$ under the map $C_{\mathrm{red}} \to A_{\mathrm{red}}$ induced by ϕ. From our discussion of I', it is clear that $\mathrm{Coker}(I_\phi)$ is isomorphic to the cokernel of the morphism given at B by $1 + \mathfrak{m}_C \otimes \mathrm{Nil}(B) \to 1 + \mathfrak{m}_A \otimes \mathrm{Nil}(B)$. In turn this implies remark (iii).

We have exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Coker}(U_\phi) & \to & \mathrm{Coker}(H_\phi) & \to & \mathrm{Coker}(G_\phi) & \to & 0, \\ 0 & \to & \mathrm{Coker}(F_\phi) & \to & \mathrm{Coker}(G_\phi) & \to & \mathrm{Coker}(I_\phi) & \to & 0, \\ 0 & \to & \mathrm{Coker}(E_\phi) & \to & \mathrm{Coker}(F_\phi) & \to & \mathrm{Coker}(D_\phi) & \to & 0. \end{array}$$

Since $\mathrm{Coker}(U_\phi)$ is clearly pseudoadditive, part (b) of the theorem follows from these sequences and 3.1(e). □

5. Line bundles becoming trivial on pullback by a nilimmersion

If X is a k-scheme, we let $\mathbf{Pic}(X)$ denote the k-functor given by

$$B \mapsto \mathrm{Pic}(X \times_k \mathrm{Spec}(B))/\mathrm{Pic}(B).$$

Then $\mathbf{Pic}$ itself defines a functor whose source is $\langle\langle k\text{-schemes}\rangle\rangle^\circ$. If $f : X \to Y$ is a morphism of k-schemes of finite type, such that X and Y each have a rational point, then $\mathrm{Ker}[\mathbf{Pic}(f)]$ is isomorphic to the k-functor given by

$$B \mapsto \mathrm{Ker}[\mathrm{Pic}(Y \times_k \mathrm{Spec}(B)) \to \mathrm{Pic}(X \times_k \mathrm{Spec}(B))].$$

Theorem 5.1. *Let k be a field, and let X be a geometrically stable k-scheme. Let $i : X_0 \to X$ be a nilimmersion, such that the ideal sheaf $\mathcal{N}$ of X_0 in X has square zero. Then there is an exact sequence of* (*abelian group*)*-valued k-functors*

$$0 \to D \oplus I \to P \to \mathrm{Ker}[\mathbf{Pic}(i)] \to 0,$$

in which D is discrete and finitely generated, I is subnilpotent, and P is pseudo-additive.

Remarks. (a) If X is affine, $\mathrm{Ker}[\mathbf{Pic}(i)] = 0$.

(b) If X is proper over k, $D = 0$ and $I = 0$, so $\mathrm{Ker}[\mathbf{Pic}(i)] \cong P$. Also, P is additive and finite-dimensional. One way to get examples is to take k to be algebraically closed, Y to be a projective variety over k, and $\mathcal{M}$ to be a coherent

$\mathcal{O}_Y$-module with $H^1(Y,\mathcal{M}) \neq 0$. Make $\mathcal{O}_Y \oplus \mathcal{M}$ into a coherent $\mathcal{O}_Y$-algebra via the rule $\mathcal{M}^2 = 0$. Let $X = \mathbf{Spec}(\mathcal{O}_Y \oplus \mathcal{M})$, and let $i : X_{\mathrm{red}} \to X$ be the inclusion. Then $\mathrm{Ker}[\mathbf{Pic}(i)] \cong \mathbb{G}_a^\alpha$, where $\alpha = h^1(Y,\mathcal{M})$.

(c) In the non-affine, non-proper case, we have not determined exactly what can happen. In particular, we do not know if D can be nonzero. If k has characteristic zero, then $D \cong \mathbb{Z}^n$ for some n. If k has characteristic $p > 0$, then $D \cong \mathbb{Z}^n \oplus (\mathbb{Z}/p\mathbb{Z})^m$ for some n and some m.

(d) Conceivably the theorem holds without the assumption that $\mathcal{N}^2 = 0$. To prove this, one would at least have to understand $\mathrm{Coker}[\mathbf{Pic}(i)]$ in the case where $\mathcal{N}^2 = 0$, which we do not.

(e) We will find an exact sequence

$$0 \to U_1 \to U_2 \to \mathbb{G}_a^\beta \to P \to 0$$

in which U_1 and U_2 are additive, for some β. This is stronger than saying that P is pseudoadditive. Perhaps P is always additive.

Proof. We have an exact sequence of sheaves of abelian groups on X

$$0 \to \mathcal{N} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 1,$$

and thus an exact sequence of abelian groups

$$H^0(X,\mathcal{O}_X^*) \to H^0(X_0,\mathcal{O}_{X_0}^*) \to H^1(X,\mathcal{N}) \to \mathrm{Ker}[\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)] \to 0.$$

In fact, everything is functorial in k, and we thus obtain an exact sequence of (abelian group)-valued k-functors

$$\mathbf{Hom}(X,\mathbb{G}_m) \to \mathbf{Hom}(X_0,\mathbb{G}_m) \to \mathbb{G}_a^\beta \to \mathrm{Ker}[\mathbf{Pic}(X) \to \mathbf{Pic}(X_0)] \to 0,$$

where $\beta = \dim_k[H^1(X,\mathcal{N})]$. Let

$$K = \mathbf{Ker}[\mathbf{Pic}(X) \to \mathbf{Pic}(X_0)], \quad L = \mathrm{Coker}[\mathbf{Hom}(X,\mathbb{G}_m) \to \mathbf{Hom}(X_0,\mathbb{G}_m)].$$

We have an exact sequence

$$0 \to L \to \mathbb{G}_a^\beta \to K \to 0.$$

According to 4.4(b), there are exact sequences

$$\begin{array}{ccccccccc} 0 & \to & R & \to & L & \to & I & \to & 0, \\ 0 & \to & P' & \to & R & \to & Q & \to & 0, \end{array}$$

in which Q is discrete and finitely generated, P' is additive, and I is subnilpotent. Let $P = \mathrm{Coker}[P' \to \mathbb{G}_a^\beta]$, which is pseudoadditive by definition. Then by 3.1(f), we have an exact sequence

$$0 \to Q \times I \to P \to K \to 0. \quad \square$$

Corollary 5.2. *Let k be a field, and let X be a geometrically stable k-scheme. Let $i : X_0 \to X$ be a nilimmersion. Let $F = \mathrm{Ker}[\mathbf{Pic}(i)]$. Let $p : B \to C$ be a faithfully flat homomorphism of k-algebras.*

(a) *If B and C are reduced, then the canonical map $F(B) \to F(C)$ is injective.*

(b) *Assume that k has characteristic zero and that the ideal sheaf of X_0 in X has square zero. Assume that B is normal and that C is étale over B. Then $\Psi_{F,p}$ (see §1, convention* (b)) *is bijective.*

Proof. Let $\mathcal{N}$ be the ideal sheaf of X_0 in X. First suppose that $\mathcal{N}^2 = 0$. Then (a) follows from (5.1). For (b), let $S = \mathrm{Spec}(B)$. Since $\mathrm{char}(k) = 0$, D is torsion-free. In

$$\langle\langle\text{(abelian group)-valued } k\text{-functors which are sheaves for the étale topology}\rangle\rangle,$$

consider the following exact sequence:

$$0 \to \mathbb{Z}^n \to \mathbb{G}_a^\alpha \to (\mathbb{G}_a^\alpha/\mathbb{Z}^n)^+ \to 0$$

arising from (5.1). (Here we let $(\mathbb{G}_a^\alpha/\mathbb{Z}^n)^+$ denote the quotient in this category.) It suffices to show that this sequence is exact when evaluated at B. Since $H^1(S_{\text{ét}}, \mathbb{Z}) = 0$ by ([1], 3.6(ii)), we are done.

Now we prove the general case of (a). For each m, let X_m be the closed subscheme of X defined by $\mathcal{N}^m$. Choose $n \in \mathbb{N}$ so that $\mathcal{N}^n = 0$. Let $K_m = \mathrm{Ker}[\mathbf{Pic}(X_{m+1}) \to \mathbf{Pic}(X_m)]$. Let $F_m = \mathrm{Ker}[\mathbf{Pic}(X_m) \to \mathbf{Pic}(X_0)]$. We have an exact sequence

$$0 \to K_m \to F_{m+1} \to F_m.$$

Then $K_m(B) \to K_m(C)$ is injective by (5.1). By induction on m, it follows that $F_m(B) \to F_m(C)$ is injective for all m. Taking $m = n$, we get (a). □

Problem. Is the functor F a sheaf?

References

[1] M. Artin, *Faisceaux constructibles, cohomologie d'un courbe algèbrique*, Exposé IX in Séminaire de Géométrie Algébrique (SGA 4), Lecture Notes in Math., vol. 305, Springer-Verlag, New York, 1973, pp. 1–42. MR **50:**7132

[2] ———, *The implicit function theorem in algebraic geometry*, Algebraic Geometry (Bombay Colloquium, 1968), Oxford Univ. Press, 1969, pp. 13–34. MR **41:**6847

[3] J. E. Bertin, *Généralités sur les préschémas en groupes*, Exposé $\mathrm{VI_B}$ in Séminaire de Géométrie Algébrique (SGA 3), Lecture Notes in Math., vol. 151, Springer-Verlag, New York, 1970, pp. 318–410. MR **43:**223a

[4] P. Gabriel, *Generalites sur les groupes algebriques*, Exposé $\mathrm{VI_A}$ in Séminaire de Géométrie Algébrique (SGA 3), Lecture Notes in Math., vol. 151, Springer-Verlag, New York, 1970, pp. 287–317. MR **43:**223a

[5] A. Grothendieck and J. A. Dieudonné, *Eléments de géométrie algébrique* IV (part 2), Inst. Hautes Études Sci. Publ. Math. **24** (1965). MR **33:**7330

[6] ———, *Eléments de géométrie algébrique.* I, Springer-Verlag, New York, 1971.

[7] R. Guralnick, D. B. Jaffe, W. Raskind and R. Wiegand, *On the Picard group: torsion and the kernel induced by a faithfully flat map*, J. of Algebra (to appear).

[8] D. B. Jaffe, *On sections of commutative group schemes*, Compositio Math. **80** (1991), 171–196. MR **92j:**14057

[9] B. Mitchell, *Theory of categories*, Academic Press, New York, 1965. MR **34:**2647

[10] M. Raynaud, *Groupes algébriques unipotents. Extensions entre groupes unipotents et groupes de type multiplicatif*, Exposé XVII in Séminaire de Géométrie Algébrique (SGA 3), Lecture Notes in Math., vol. 152, Springer-Verlag, New York, 1970, pp. 532–631. MR **43:**223b

[11] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443. MR **18:**514a

[12] J-P. Serre, *Algebraic groups and class fields*, Springer-Verlag, New York, 1988. MR **88i:**14041

[13] W. C. Waterhouse, *Introduction to affine group schemes*, Springer-Verlag, New York, 1979. MR **82i:**01034

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